Objectification of upper-secondary teachers' verbal discourse in relation to symbolic expressions

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Abstract Research literature points to the importance of objectification when learning mathematics, and thereby in the discourse of mathematics. To increase the field’s understanding of aspects and degrees of objectification in various mathematical discourses, our study uses the combination of two sub-processes of objectification in order to analyse upper-secondary teachers’ word use in relation to any type of mathematical symbols. Our results show that the verbal discourse around symbols is very objectified. This can put high demands on students understanding of their teacher, since it might be needed that the students have reached a certain degree of objectification in their own thinking in order to be able to participate in a more objectified discourse. The results also show that there exist patterns in the variation of the degree of objectification, in particular that the discourse tends to be more objectified when more familiar symbols are used. This exploratory study also reveals several phenomena that could be the focus of more in-depth analyses in future studies.

Keywords Alienation, Mathematical learning, Mathematical symbols, Natural language, Reification, Word use

1 Introduction

The duality of mathematical concepts, as both processes and objects, and how students' ability to perceive both perspectives (i.e., to objectify concepts) is essential for a deep mathematical understanding, and has been discussed by several researchers from various perspectives over the years (e.g., Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991, 2008). Symbols are used to represent abstract mathematical concepts, and play a vital role in students' objectification of concepts (Caspi & Sfard, 2012). Furthermore, it is widely recognized that the mathematical discourse in the classroom is vital for students' learning (e.g., Ben-Yehuda, Lavy, Linchevski & Sfard, 2005; Radford, 2000; Österholm, 2012). Thus, how symbols are addressed in the discourse, particularly through the used words (Sfard, 2008), is of great interest in order to understand knowledge development.

This paper contributes by empirical analyses of upper-secondary teachers' mathematical instruction, focusing on their word use in relation to mathematical symbols. We search for patterns in the discourse, with the purpose to understand what aspects and degrees of objectification there are in various situations and for different kinds of mathematical symbols.

2 Characterizing mathematical discourse

In this study, we focus on the discourse of mathematics classrooms, and particularly teachers' discourse. There is complexity in the analyses of discourse, for example, Gee (1996) distinguishes between Discourse (with capital "D") and discourse (with lowercase "d"), where *Discourse* is a distinct way to use language integrated with ways of for example believing, dressing, and using various types of artefacts, and *discourse* is the language that is in use. The study of Discourse also includes a focus on the social functions of language, including aspects of positioning and power relations (e.g., see Herbel-Eisenmann & Wagner, 2010). In our analyses, we focus on language use in relation to mathematical aspects, which is one part of the mathematics classroom Discourse. Our focus is not on the social functions of language, although these are always present in the background. As highlighted in a comprehensive review of discourse research in mathematics education (Ryve, 2011), in mathematics classrooms there are of course important more *generic educational discourses* that do not have any focus on the mathematical, for example, the shaping of identity positions or aspects of positioning and power relations. The review showed that it is equally common for research studies in mathematics education to focus on mathematical aspects of discourse or to focus on more generic educational discourses. Our study primarily contributes to the area of research where focus is on mathematical aspects of discourse.

To distinguish what makes a discourse mathematical, Sfard (2008) focused on word use, visual mediators, endorsed narratives, and routines. We focus on two of these parts: word use and the use of mathematical symbols (as part of visual mediators). We delimit our study to these two parts of discourse since these are essential for issues of objectification (as discussed below) and in order to be able to make more in-depth analysis of relations between these parts of language use in mathematics classrooms. Furthermore, these two parts can be seen as two distinct languages that are used in mathematics, since we define language as "a system, used for communication, comprising finite set of arbitrary symbols and a set of rules (or grammar) by which the manipulation of these symbols is governed" (Sfard, 2008, p. 101). That is, in mathematics, different languages are used. One is a natural language, that is, "a language that is the native speech of a people" (Dictionary by Merriam-Webster), such as English or Swedish. Basic components of a natural language are sounds (phonemes), words, and sentences. Besides more colloquial words, natural language also includes specific technical words (e.g., "differentiable" and "addition") and particular grammatical constructions (e.g., "twice as many as"). Another language in mathematics is the highly specialized symbolic language, used to express such things as numbers, equations, functions, and formulas (e.g., "125", "x + 1/x = 2", and "$f(x) = e^x + 3$"). The symbolic language can be distinguished from natural language since the symbolic language is a written language and does not build on sounds or on the written coding of sounds (as natural language does).

A key part of knowing and doing mathematics is being able to communicate mathematics using the different types of languages that constitute the mathematics discourse. Much research points to the necessity of relating the different languages to each other, especially natural language and the symbolic language of mathematics. For example, such relations include describing the meaning of mathematical symbols, being able to translate between natural language and symbolic language, and to use mathematical symbols that are parts of a text, that is, where the symbols add meaning in the same way as the surrounding natural language (see Arcavi, 1994, 2005). Furthermore, empirical studies showed that it is common for students not to handle mathematical symbols in a relational way when symbols are mixed with natural language, which could result in problems of understanding (Österholm, 2006). Other empirical studies showed the importance of the natural language as a starting point for the development of different aspects of knowledge of the symbolic language (Caspi & Sfard, 2012; Redden, 1996), and several scholars discuss how it is necessary with linguistically-oriented instruction for the development of meaningful understanding of symbolic language (Drouhard & Teppo, 2004; Meaney, 2005; O'Halloran, 2008). Thus, our study contributes to this line of research by making in-depth analyses of relations between the use of natural language and the use of the mathematical symbolic language.

Besides the importance of using the different languages and relating them to each other, as addressed above, it is also important *how* the languages are used and related to each other. Objectification is an important aspect of how languages are used, in relation to the teaching and learning of mathematics, since a core of mathematics is the development of abstract concepts, which can only be created through a development in the discourse. In particular, the use of symbols plays a central role in this process of objectification (Caspi & Sfard, 2012), but the natural language also has an important mediating function in objectification (Park, 2016). Therefore, our in-depth analyses of relations between the use of natural language and the use of the mathematical symbolic language focuses on issues of objectification in the discourse.

2.1 Objectification in mathematical discourse

A widely recognized difficulty in the learning of mathematics is the inherent process-object duality of mathematical concepts, which has been addressed through various notions over the years (e.g., Dubinsky, 1991; Grey & Tall, 1994; Sfard, 1991, 2008; Tall, 2008; Tall & Vinner, 1981). For example, Sfard (1991) distinguished between *operational* and *structural* conceptions, where *operational* refers to processes related to a mathematical concept, and *structural* refers to treating a mathematical concept as some abstract object. She furthermore claimed that the operational conception precedes the structural, and defined *reification* as the ontological shift that occurs when a person becomes able to see a notion previously tightly connected to a process, as a fully-fledged object. *Encapsulation* is another notion, very similar to reification, as used by Dubinsky (1991), to describe the process when mathematical concepts, first conceived as dynamic processes, are converted into static objects. Although encapsulation is regarded fundamental in order to provide meaning to the concept, Dubinsky stressed the importance to be able to alternate between process-object representations of a concept. A similar requirement for success in mathematics is addressed by Gray and Tall (1994) when they highlighted how flexibility to think about notations for mathematical concepts both as processes and objects "is the root of successful mathematical thinking" (p. 5). They introduced the term *procept*, which emphasizes the importance to embrace the duality of a symbol as both the process and the product of the process. All these previous studies have been conducted from a cognitive perspective, which treats the individual and the social as separate units and thus focusing on the individual's learning of mathematical concepts. By instead taking a discursive perspective, which treats thinking as intrapersonal communication, individuals and the social are not seen as separated units, instead it is the activities that are in focus and become the units of analysis (Sfard, 2008). Learning is then considered to be the development of the individual's discourses, and *reification* was re-defined to denote when processes are discursively turned into objects. In this study, we adopt such a discursive perspective on mathematics and the learning of mathematics, in line with Sfard (2008).

From this perspective, a core question for mathematics education research is if and how certain properties of classroom discourse can help students to develop their own mathematical discourse. Twelve years ago (Hiebert & Grouws, 2007), this type of research was described as *emerging* in a review of research on the effects of classroom mathematics teaching on students' learning, but there were still no clear empirical links between particular kinds of discourse and students' learning. More recently, Sfard (2016) noted that "there are still more questions than answers about the relation between mathematics teachers' teaching and their students' learning" (p. 41). To be able to approach the core question, it needs to be more specific, which in this study is done by focusing on issues of objectification in the discourse. Furthermore, which we will see through previous research on discourse of objectification, when approaching the question of relations between teaching and learning, we also need more research about properties and patterns in the discourse of objectification. This study contributes with such research, by focusing on the discourse of mathematics teachers.

Sfard (2016) described two basic ways for teachers to support students to develop their mathematics discourse. One way is to explicitly encourage the desired kind of discourse, which has been examined by Güçler (2016). Another way is to model the discourse they want their students to develop, which is the focus of our study. This aspect of teaching has been examined not only from discursive perspectives (e.g., Güçler 2013, 2014; Sfard, 2016), but for example also concerning mathematical reasoning (Bergqvist & Lithner, 2012) and teachers' lecturing in general (Rodd, 2003). Of course, the teacher needs to adapt her discourse to fit the students, based on the present status of the students' discourse, to promote a gradual change in this discourse. But how do teachers adapt their discourse when it comes to issues of objectification? Our study addresses this question by using an analytic framework for discourse of objectification, based on the theory of Sfard (2008), where two sub-processes of objectification are examined in combination, which we have not seen in previous empirical studies based on the theory of Sfard. This framework is used to explore variation and patterns in teachers' discourse of objectification, where variation and patterns concern how type and degree of objectification exist in relation to different types of symbols and situations.

Previous empirical research on objectification in mathematical discourse addresses issues of student learning, of relations between teaching and learning, and to some extent of variation in the discourse of teaching. These three lines of research are discussed below.

Research points to the importance of objectification for student learning, for example, through empirical results showing that students whose arithmetic discourse was objectified were likely to use the discourse on their own, without involvement by others, and thus show a higher degree of understanding than those whose discourse was not reified (Ben-Yehuda et al., 2005). Research also shows that objectification is a complex learning process, for example, by a study following students during two months of teaching about functions, where objectification "had barely begun" after these two months (Nachlieli & Tabach, 2012, p. 24). However, in some situations, young students have shown to spontaneously develop a verbal reified discourse about numbers and operations before they had any formal schooling in algebra (Caspi & Sfard, 2012). These types of results highlight the need to address questions about the relations between teaching and learning.

Several studies examine relations between teaching and learning by comparing the discourse of students with the discourse of their teacher. These studies show mixed results. Although the teacher's discourse on limits was dominantly objectified, one study showed that the students did not objectify the concept (Güçler, 2013). Further analyses in the same study showed that a difference was found between the students' and the instructor's discourses (Güçler, 2014). The instructor clearly distinguished between the process and product aspects of the limit notion, whereas this distinction seemed unnoticed by the students, as they mostly used words denoting operations/processes when talking about limits. Results from Emre-Akdoğan, Güçler and Argün (2018) showed a variety of connections between teacher discourse and student discourse. The discourse of some students was more similar to the teacher discourse and the similarity varied depending on which part of the discourse was compared, such as word use or endorsed narratives. Sfard (2016) on the other hand showed clear similarities between teacher discourse and student discourse on objectification, but this similarity concerned a *lack* of objectification. In an experimental study, students were exposed to different types of presentations, where variation in the presentations concerned the degree of objectification in the discourse (Österholm, 2012). The results showed that when students were asked to summarize the presentations, their discourse was generally more process-oriented than objectified, regardless of the degree of objectification in the presentation. However, students' summaries of the more objectified presentation were somewhat more objectified compared to their summaries of the less objectified presentations, but this effect was far smaller than differences between individual students. In total, empirical studies show that the effect of teacher discourse on the development of student discourse is possible but tend to be small – at least in the short time frames that have been examined in these studies. Some aspects of variation in teacher discourse are mentioned in these studies, but the studies do not give much information about such variation since each study focuses on a singular case or an experiment. However, some studies allow for more specific information about variation of objectification in teacher discourse.

Nachlieli and Tabach (2012) described several different types of pedagogical activities and principles that teachers use in relation to issues of objectification, when teaching about functions. Other studies focused on properties of discourse, which is more relevant for the purpose of our study. When analysing the teaching of limits, Güçler (2013) noticed how the teacher in some situations, when addressing an informal definition and when computing limits, shifted between focusing on limit as a number and as a process, while the discourse in other contexts was more consistent. Kenneman (2014) analysed both teachers' algebraic discourse (i.e., unknown numbers, magnitudes and quantities) and the algebraic discourse in mathematics textbooks for compulsory school. She concluded that there was an increasing occurrence of objectification in natural language used in the textbooks from grade 2 to 8, mostly between grade 2 and 5, and to some extent between grade 5 and 8. Furthermore, she showed that the degree of objectification in teachers' discourse in grade 8 was lower than the degree of objectification in corresponding textbooks. The teachers' discourse more resembled that of grade 5 textbooks. Other studies have not examined the teacher, but focused only on textbooks, concerning properties in the discourse of objectification. Although using different methods, Österholm and Bergqvist (2013) showed a similar result as Kenneman (2014), concerning an increasing degree of objectification (through an increasing amount of nominalizations and use of nouns compared to verbs) in textbooks from grade 4 to 10, where the increase was mostly between grade 7 and 10. Objectification in textbooks was also investigated by Park (2016), but her focus was on a few aspects of the derivative concept at university level. She concluded that words have a large role in mediating reification, compared to visual mediators, and that words are crucial in order to connect different visual mediators (e.g., mathematical symbols and graphs) used to, for example, objectify derivative at a point.

In summary, concerning the empirical studies discussed above, the importance of objectification for students' learning is evident, but the process of objectification is very complex and difficult to achieve. Teachers can use a variety of methods to help students develop their discourse concerning objectification, where the modelling of wanted properties of discourse is one such method. However, we know very little about which ways of modelling do allow students to develop their own discourse. That is, we do not know how teachers vary their discourse concerning objectification. Our study contributes with information about such variations, concerning how type and degree of objectification exist in relation to different types of symbols and situations, which is needed for several reasons, as described below.

Most of the empirical studies discussed above focused on specific mathematical content, for example algebra, or specific mathematical symbolic expressions, for example the limit notation. However, objectification is a general phenomenon, applicable on any mathematical concept. Therefore, it is of interest to examine if and how there are any patterns in objectification in mathematical discourse over different types of content. Information about any such patterns could be used in future studies of students' learning, for example, to examine if and how students who have more experiences of objectification of mathematical concepts more easily can develop an objectified discourse for a new concept. This type of research is scarce, as shown in a review of research in the field of language and communication in mathematics education: "little attention has been given to the more general issue of the acquisition of mathematical ways of speaking or writing that may be applicable and acceptable in a wide range of areas of mathematics" (Morgan et al., 2014, p. 851).

Furthermore, a literature review showed that there are many claims in research literature about properties of mathematical discourse, but not many empirical studies exist that in a structured manner characterize mathematical discourse concerning important/central mathematical properties (Österholm & Bergqvist, 2013). For example, the review showed that it is common to claim that mathematical discourse is compact or dense, which often is related to the presence of many nominalizations or long, complex noun phrases, which in turn can be related to issues of objectification. The review also showed that claims about the compactness of mathematical discourse are seldom explained in more detail and are often given as a characterization of mathematics in general. However, we know from empirical studies (discussed above) that the degree of objectification varies over school years, and it is thus unclear if or how a general characterization of mathematical discourse is meaningful. Therefore, we need more empirical research on properties of mathematical discourse. In particular, we need studies that examine what types of variation exist in mathematical discourse, for example, over different grade levels, different teachers, different content areas, etc.

In this study, we focus on objectification in teachers' mathematical discourse, to examine variations in how they model certain properties of mathematical discourse, which can give students different opportunities to learn mathematics. The type of variation we focus on concerns how type and degree of objectification exist in relation to different types of symbols and situations. Furthermore, we do not focus on a specific mathematical concept, but how the objectification is mediated by natural language and mathematical symbols more generally.

3 Theoretical Framing

This study uses Sfard's (2008) definition of objectification and its two sub-processes, reification and alienation. *Objectification* is regarded as a "process in which a noun begins to be used as if it signified an extradiscursive, self-sustained entity (object), independent of human agency" (Sfard, 2008, p. 300). The process of objectification is divided into two tightly related sub-processes: *reification* – "the act of replacing sentences about processes and actions with propositions about states and objects" (p. 44), and *alienation* – the process when "the alleged products of the mind's actions may undergo the final objectification by being fully dissociated, or alienated, from the actor" (p. 50). Ontologically, this implies that objects do not exist "objectively", but are the sum of ways of communicating about them. As an example of reification, compare the sentence "In Newton's theory, the word 'force' *was used differently* than in the Aristotelian physics" (p. 44), with a reified version "The word 'force' *had a different meaning* in the Newtonian and Aristotelian theories" (p. 44). As an example of alienation, compare the sentence "***We shall call*** *a polygon a triangle if and only if it has three sides*" (p. 57) with an alienated version "*A polygon* ***is*** *a triangle if and only if it has three sides*" (p. 57). By taking this theoretical standpoint, learning mathematics occurs by developing mathematical ways of using language, in particular through reification and alienation.

Reification and alienation are parts of the larger work to develop a theoretical framework for a communicational approach to cognition. The framework is developed specifically to highlight the mathematical aspects of a discourse, that is, to be able to see what it is that makes the discourse mathematical. Therefore, this framework is suitable for the purpose of this paper, where focus is on the mathematical aspects of discourse. In this paper, we use two pairs of notions that characterize mathematical discourse, concerning word use in relation to objectification. In relation to the process of reification, we characterize discourse as being *process-oriented*, when focusing on processes and actions, or *object-oriented*, when focusing on states and objects. Similar categorisation of word use in relation to Sfard's (2008) theory has been used in previous studies (e.g., Güçler, 2014). Furthermore, in relation to the process of alienation, the terms personified and alienated are used to characterise discourse about both mathematical processes and mathematical objects. Discourse that is *personified* implies that there is a person present that either is involved in the process or is an owner of an object. *Alienated* discourse on the other hand is used about processes that occur without human interference or about objects that exist irrespectively of a human agent. Since we address mathematical aspects of discourse, and not more general educational discourse (cf. Ryve, 2011), we focus on the existence or non-existence of human agents in the discourse. This allows us to capture if the mathematics, as described in relation to the symbols, exists and acts independently of human beings. Considerations of *how* human beings are used in discourse is important when studying the "whole" mathematical classroom Discourse (Gee, 1996) or the more generic educational discourse (Ryve, 2011). For example, different uses of "we" can relate to power positioning and identity shaping as discussed in research focusing on generic educational discourse (e.g., Herbel-Eisenmann, Wagner, & Cortes, 2010; Rowland, 1999).

Sfard (2008) explicated that, although tightly related, the two processes of reification and alienation do not necessitate each other, and the "two types of transformation are attained by different means" (p. 44). That is, the two processes can occur independently, thus creating four possible types of instances of discourse in relation to objectification, as seen in Table 1, where reification is movement to the right in the table, while alienation is movement downwards. This specifies the degree of objectification by saying that an instance of discourse is more objectified if it is to the right or below another instance of discourse.

Table 1 Examples of how different ways of using natural language in relation to the symbolic expression 3 + 4 = 7 results in different degrees of objectification of the mathematical discourse.

3 + 4 = 7	Process-oriented	Object-oriented
Personified	If we add three and four we get seven	We have three plus four, which is equal to seven
Alienated	Three plus four becomes seven	Three plus four is equal to seven

Mathematical symbols have been created specifically to serve as visual mediators in the mathematical discourse (Sfard, 2008). This symbolization could be regarded as a generalization and contributes to prevent ambiguity, to standardize the discourse, and to make expressions more compressed (Caspi & Sfard, 2012). The symbolization of a mathematical discourse reinforces the effect of reification of the discourse (Caspi & Sfard, 2012), thus mathematical symbols are important to accomplish objectification. To become a fully-fledged participant of a mathematical discourse, alienated descriptions about mathematical objects and the relations between objects are to be used. Thus, objectification plays a central role in students' mathematical knowledge development (Caspi & Sfard, 2012).

4 Purpose and Research Questions

The purpose of this study is to increase the field's understanding of aspects and degrees of objectification in various mathematical discourses, which is an addressed need (as discussed above). This purpose is fulfilled by empirically analysing and characterising teachers' use of natural language in relation to their use of mathematical symbols. There are no students involved in this study, and thus no analyses of actual learning. However, the study of teachers' mathematical discourse gives valuable and important insights about which opportunities for learning that are provided to students. As described above, complete objectification is reached when the discourse focuses on objects, and not processes, in an alienated manner. Thus, the focus of the analyses is on the process-object distinction and on degree of alienation. The degree of objectification (how objectified the discourse is) depends both on the reification, by replacing talk about processes with nouns, and on the alienation, by disconnecting from the human actor. We delimit our study to the analyses of verbal word use and written mathematical symbols, since these are two central parts in a mathematical discourse, in particular for objectification (Caspi & Sfard, 2012; Sfard, 2008). We analyse the relation between these two parts, concerning objectification in various mathematics discourses, covering different mathematical topics, and not with respect to objectification of a specific concept or symbol. This type of analyses allows us to explore if there seem to exist any general patterns in ways of expressing oneself in relation to different kinds of mathematical symbols. More specifically, the following research questions are addressed:

> RQ1: To what degree is teacher-talk about symbolic expressions mainly process- or object-oriented? And to what degree is the talk mainly personified or alienated?
>
> RQ2: With respect to the differentiation of teacher-talk in RQ1, what types of patterns are there of when and how objectification occurs in teachers' discourse?

Our primary focus is RQ2, where we search for patterns in the discourse. However, we also include RQ1, which is a basis for RQ2, but where we also make a general characterization of the degree and type of objectification in upper secondary teachers' mathematical discourse. That is, RQ1 produces an answer that gives a quantitative distribution of teacher-talk in four categories, through the combination of process- and object-oriented on the one hand, and personified and alienated on the other hand. This distribution describes if and to what degree objectification primarily occurs through reification or alienation. This result is thereby part of the general purpose of characterizing the objectification in mathematical discourse, which in this study is delimited to teachers' discourse in upper secondary school. These results from RQ1 can be used to compare with previous and future analyses of other types of discourse, to reach a better characterization of variations in mathematical discourse.

The answer to RQ2, on the other hand, focuses on a description of any type of pattern *within* the four categories mentioned above. This description produces another type of characterization of how objectification occurs in mathematical discourse. In particular, we deliberately cover different mathematical topics, and do not limit the study with respect to objectification of a specific concept or symbol. This is done to be able to explore variation concerning degree or type of objectification in relation to variation of type of symbol or type of situation. Thus, the patterns of *when and how* refer to whether a certain degree or type of objectification occurs for certain types of symbols or types of situations.

In general, the research questions in this study focus on characterizations of mathematical discourse. Such a characterization can be done at a very general level, for example, as Sfard (2008, p. 129) did when focusing on what makes mathematical discourse distinct, and can also be done at a very local level, for example, as Sfard (2008, p. 128) did when focusing on patterns in an individual person's discursive actions. Our study is located somewhere between these extremes. We do not focus on the individual, but we delimit our study to the community of mathematics teachers at upper secondary school, where we search for patterns concerning when and how objectification occurs in situations when the teachers talk about symbolic expressions. We argue that it is reasonable to characterize, and find patterns in, the discourse among mathematics teachers at upper secondary school, at least in the Swedish context, which is examined in this study. In Sweden, upper secondary school is a specific form of education (cf. Swedish National Agency for Education, 2013), for which teachers have a specific type of teacher education. In summary, the focus of this study is a characterization of the discourse of mathematics teachers at upper secondary school, especially concerning patterns regarding objectification when teachers talk about symbols, as described above, which will produce deeper understanding of aspects and degrees of objectification in mathematical discourse.

5 Method

5.1 Research design

We are not certain beforehand if, how clearly, or what types of patterns regarding objectification that can be found. In this sense, the study is exploratory and of more descriptive type. In addition, the study does not focus purely on qualitative issues nor purely on quantitative issues. That is, there is a balance between breadth and depth in the analyses. The breadth concerns variation in the types of situations that are analysed, but where a broader context of each situation is not analysed. This balance in breadth is chosen to be able to make more in-depth analyses concerning specific properties of the discourse, concerning how verbal natural language is used in the objectification of mathematical symbols. In particular, the data in this study comes from seven teachers, who altogether produce many instances (260) that are analysed concerning aspects of objectification. The large number of instances is suitable as a basis for finding patterns concerning specific properties of the instances, but where no in-depth analyses are made concerning explanations of such properties, which would need a more contextualized type of analysis for each instance, which is suitable for a smaller selection of instances. In addition, the seven teachers, who are chosen through a random selection, allow for an analysis of a variation of situations when mathematical symbols are used, which is relevant for a characterization of the discourse among teachers at this school level. However, the selection is too small to make any direct generalizations.

This type of design is used to be able to find different types of phenomenon or patterns that can be examined in more detail in future studies. For example, certain phenomenon found in this study could be analysed more in-depth in future studies that focus on qualitative and explanatory aspects where a broader context is considered. In addition, any patterns found in this study could be examined in more large-scale studies where the limits of generalizability could be put to the test, including if and how these patterns are similar or different at other school levels. Any patterns that are found in this study need to be interpreted to become more meaningful, which is done in the discussion section, where we also highlight certain types of future studies that could develop the findings from this study.

5.2 Data

Data consist of voice recordings of seven upper secondary teachers, done at one of their ordinary mathematics lessons. The recordings are from a larger study, the year before the teachers entered a competence development program for mathematics teacher. The recordings were used as part of an evaluation of that program, for which 51 teachers were randomly selected all over Sweden, but the teachers also agreed that the recordings could be used for research purposes. The seven teachers in the present study were then randomly chosen among the 51 participating upper secondary teachers. In addition to the voice recordings, there are photos of the whiteboards and/or notes done by the observer of what the teachers wrote on the whiteboards. Swedish upper secondary school is a voluntary continuation of compulsory school for students between 16-20 years. There are 18 national programmes, each three years long, which is either vocational or as preparation for university studies. In each programme, there is at least one compulsory mathematics course. A compilation of available information about the lessons are displayed in Table 2. As seen in this table, there is a variation in the mathematical content, which is a prerequisite for our purpose.

Table 2 Information about the teachers' lessons.

Teacher	Programme/Grade/Number of students	Length of lesson/ Analysed part (minutes)	Mathematical Content
1	Natural Science/1/26	70/30	Graphical representations of straight lines and linear system of equations
2	Electricity and Energy/2/12	65/25	Equivalence and implication
3	Natural Science/1/29	70/37	Logarithmic relations and logarithmic equations
4	Technology/1/14	80/40	Complex numbers and quadratic equations with complex roots
5	Technology/1/unknown	70/30	Quadratic equations
6	Technology/1/30	70/15	Quadratic equations
7	Technology/3/23	50/13	Complex numbers

The recordings of teachers' spoken words were transcribed and then coordinated with the notes and photos of teachers' written mathematical symbols. Since our focus is on teachers' discourse, with respect to their verbal word use in relation to their written symbols, and data about the used symbols only were available for the parts of lessons when the teachers spoke to the whole class, we only used these parts in our analysis. The different teachers' modes of teaching in the classroom had some variations, which affected the amount of data included from each teacher, among other, reflected in the time-differences of the analysed parts (see Table 2).

5.3 Analysis procedure for objectification

The overall focus of our analyses is to characterize teachers' mathematical discourse, in particular their word use when talking about mathematical symbols, concerning the degree of objectification in this discourse. Our study is delimited to how verbal natural language is used in relation to mathematical symbols. Therefore, we focus on teachers' word use, and do not include other aspects that are present in the discourse that may influence reification of mathematical concepts, such as teachers' gestures.

We categorise the word use in four categories, created through combinations of process- or object-oriented and personified or alienated (see theoretical framing, particularly Table 1). This process of categorisation is described below. We use this categorisation to answer our research questions in the following way. First, we examine how teachers' utterances are distributed over the four categories, to examine to what extent the discourse is objectified overall. That is, we examine if the discourse is more object- or process-oriented and if it is more alienated or personified. Second, and which is our main focus, we characterise the types of symbols and situations that are present in the four categories. This will allow us to explore if there are any general patterns in the ways teachers express themselves in relation to certain kinds of mathematical symbols or situations. That is, this part of the analysis focuses on finding themes within the four different categories. To keep the study focused on the discourse of objectification and the relation between teachers' talk and the symbolic expressions, the themes we search for primarily address any patterns concerning how teachers talk about the symbols. That is, our analyses focus on the specific utterances from teachers; what they are talking about and how they address the symbols. We do not focus on broader characterisations of the situations where the utterances occur, such as what type of task is used or general properties of the whole symbolic expression that have been written on the whiteboard. For example, in many situations, the teachers write equalities on the whiteboard, but sometimes they do not talk about the equality itself, but some part of the whole symbolic expression. In such situations, we focus on the part of the symbolic expression that is directly addressed by the teachers' talk, and we use this as a basis for finding patterns in the different categories of objectification.

The first step in analysing degrees of objectification of teachers' oral presentation in relation to written symbolic expressions is to identify utterances in the transcripts where teachers spoke about symbolic expressions, and secondly to characterize the words they used. We identify utterances for analyses through a two-step procedure. First, we delimit what we consider as one symbolic expression, which is done based on the behaviour of the teacher. In particular, one symbolic expression is the symbols that a teacher writes without distinct pauses, but where a distinct pause exists when the teacher stops writing and starts talking about the symbols that have been written or starts talking about something else. There are no formal definitions of what constitutes a symbolic mathematical expression, and this way of identifying a symbolic expression is in agreement with our theoretical point of view, that mathematics is constructed in the discourse. Therefore, we do not use any external criteria for what should be considered a symbolic expression. Second, we decide which symbolic expression a teacher talks about by focusing on both explicit and more implicit signals by the teacher. Explicit signals are when the teacher talks while writing the symbols or when directly referring to the symbols through words (e.g., "we see here…"). Implicit signals are when the content of the talk is clearly related to the content of the symbolic expressions (see 5.3.1 for examples). As seen in the results, we consider both the identified symbolic expressions, as well as the constituent symbols of the expressions in the analysis.

When characterizing the words teachers use in their talk concerning objectification, we rely on the definition from Sfard (2008), as described above in the theoretical framing. Their word use is considered process-oriented if they use active verbs and thus categorised as *process-talk*. If instead nouns and static relations are focused on when talking about the symbolic expression, the word use is categorised as *object-talk*. The degree of alienation in the discourse depends on the presence of human agency; if mathematical objects are considered independent agents or not. Since we are interested in the presence of any human interaction at all, teachers' talk is categorised as *personified* if a personal pronoun is used or if there is passive voice in the talk, otherwise the talk is categorised as alienated (see Table 1 for example). The categorisation procedure was developed and refined together by the two authors through analyses of some of the teachers' talk in relation to the written symbolic expression. This continued until consensus of the procedure was reached. When all data had been analysed, we discussed situations that were difficult to categorize and agreed on principles of how to categorize these.

5.3.1 Examples

The following symbolic expressions and associated utterances are chosen to exemplify the categorisation procedure and to illustrate the different types of analysis. The original language of the utterances is Swedish, displayed in parentheses. During translation to English, corresponding mathematical verbal expressions used in relation to mathematics education were considered.

<u>Personified process-talk:</u>
Symbolic expression: $0.84^x = 0.25$ becomes $\lg 0.84^x = \lg 0.25$
Utterance: "we take the logarithm of both sides" ("vi logaritmerar båda sidorna")

Here the teacher was solving an equation on the whiteboard and expressed the next step by using the active verb *take*. By using the personal pronoun *we,* there was a human action involved. The talk around the symbolic expression $\lg 0.84^x = \lg 0.25$ was thus considered processual, and dependent on what the person did. This categorisation does not apply to the expression $0.84^x = 0.25$, which was characterised separately.

<u>Alienated process-talk:</u>
Symbolic expression: $\begin{cases} x_1 = 1 + 2i \\ x_2 = 1 - 2i \end{cases}$
Utterance: "the solutions to this equation become x-one equals one plus two i and x-two is equal to one minus two i" ("lösningarna till den här ekvationen blir x-ett lika med ett plus två i och x-två är lika med ett minus två i")

When solving a complex equation, the teacher expressed the solutions as something that *become* (active verb) two different values of x. As the example above, the talk around the symbolic expression was considered processual, but in this case without any human interaction and thus alienated. Since the teacher wrote the solutions joined with a curly bracket without any paus in the writing and talking (i.e., both solutions were addressed at the same time) this was considered as one symbolic expression and not two. This categorisation does not apply to the including component of the expression, like the equal sign, which was characterised separately.

Personified object-talk:
Symbolic expression: a · b
Utterance: “because a multiplied by b, we could, of course, see it as a single number” (“därför att a gånger b, vi skulle ju kunna se det som ett enda tal”)

In this sentence, it was explicated that the product could be considered to be an object, *a single number*, but this depended how *we* choose to see it, that is, the object was connected to a person.

Alienated object-talk:
Symbolic expression: y = 3 – kx
Utterance: “the m-value is three, for that line the m-value is three”[1] (“m-värdet är tre, för den linjen är m-värdet tre”)

The use of *that line* was interpreted as that the whole expression was considered as an object, a line, which exists as an entity of its own, where no human actor is involved. This categorisation also holds for the including part “3, the m-value”, in the separate characterisation of the including components.

6 Results

In total, there were 199 different symbolic expressions written on the whiteboards that the teachers spoke about. The expressions differed in length and complexity. They varied from one single symbol, for example, *r* used to denote the length of a complex number, to long expressions or equalities, for example, $10^{lg(a \cdot b)} = a \cdot b = 10^{lga} \cdot 10^{lgb} = 10^{lga+lgb}$. Sometimes teachers made more than one utterance about a particular symbolic expression and its constituent parts. In total, 260 different utterances about the 199 symbolic expressions were identified. As seen in Table 2, the lengths of the analysed parts of respective lesson varied from 13 to 40 minutes, which in turn influenced the number of utterances from respective teacher. The number of utterances varied between 14 from one teacher and 67 from two of the teachers.

6.1 Descriptive statistics (RQ1)

Table 3 shows that object-talk in relation to the symbolic expressions dominated, 199 vs. 61 occasions.

Table 3 Distribution of utterances with respect to type of discourse.

	Process-talk	Object-talk	Total
Personified	43	107	150
Alienated	18	92	110
Total	61	199	260

[1] In Swedish, *m* is used to denote the constant value in the equation for a straight line

Furthermore, process-talk was more often personified than alienated, 70 % vs. 30 % of the times. If instead object-talk was used, the difference was not that distinct, 54 % were personified and 46 % were alienated (Table 3).

Mostly, the teachers' utterances for one symbolic expression were within one category (Table 4). But in relation to 49 of the 199 different symbolic expressions, the teachers made multiple utterances that were categorized differently.

Table 4 Distribution of symbolic expression with respect numbers of used discourse categories.

Symbolic expressions	One category	Two categories	Three categories	Four categories
199	150	39	8	2

As seen in Table 4, two differently categorized utterances were used in relation to 39 of the symbolic expressions, while three or four differently categorized utterances in relation to one symbolic expression were quite rare.

6.2 Characterisation of the categories (RQ2)

6.2.1 Personified process-talk

When analysing the situations when the teachers used personified process-talk in relation to a symbolic expression, it emerged that it was mostly when the equivalence between steps in a solution to a problem was described or motivated (for 27 of the 43 occasions), that is, how one symbolic expression "became" another one. No particular similarity for the rest of the occasions emerged.

> Example: One teacher wrote the expression "2kx = 3" and said "you have added kx on both sides and get two kx is equal to three". The words "add" and "get" indicates a process-talk in relation to the symbolic expression, and it is the process of how "kx = 3 – kx" becomes "2kx = 3", due to what the human actor "you" do with the expressions.
>
> A similar type of example is when a teacher had written "$x^2 + 1 = 0$" and asked the students how they would solve the equation. After suggestions the teacher said "yes, we eliminate minus one on both sides and then get x squared minus one", and the following expression was written on the whiteboard "$x^2 = -1$" (*note*. this is the teacher's utterance in relation to the expression, although it may sound wrong). A third example is given in 5.3.1.

6.2.2 Alienated process-talk

If the process-talk instead was alienated it was often the last step in a solution that was addressed (for 13 of the 18 occasions), either that the teacher concluded what the answer was or encouraged the students to give the answer. As for the above category, it could be the equivalence between the steps in a solution procedure that was intended, or it could be the equal sign that was treated like a process.

> Examples: In one instance, one teacher wanted the students to come up with an expression for how the area of a triangle depends on the slope *k* of a straight line. He had written the expression "$A_1 = \frac{\frac{3}{k}\cdot\frac{3}{2}}{2} =$ " and said "okay, what does that become?" a student gave an answer and the teacher said "nine divided by two k that then is to be divided by two, and

then it becomes?" and continued expanding the expression "$A_1 = \frac{\frac{3}{k}\cdot\frac{3}{2}}{2} = \frac{\frac{9}{2k}}{2} =$ ". The teacher summarised an answer from the students "yes, how good, nine divided by four k, so" and completed the expression "$A_1 = \frac{\frac{3}{k}\cdot\frac{3}{2}}{2} = \frac{\frac{9}{2k}}{2} = \frac{9}{4k}$". In this example the equal sign was treated as a process, since the focus was on *becoming* something, but no human was involved in this process since it was the expressions that were the actors in this *becoming*. Another example, when it was the equivalence that was processual, is given in 5.3.1.

6.2.3 Personified and alienated object-talk

We found no specific pattern that applies to only personified object-talk, but some pattern for object-talk in general (as described in this section) and some pattern for alienated object-talk (as described in the next section).

In both categories of object-talk (personified and alienated), teachers addressed whole expressions as single objects, as well as the included symbols as individual objects. This pattern thus seems to apply to more object-oriented discourses, irrespective of degree of alienation.

Examples: One teacher introduced the concepts of implication and equivalence, and treated the symbols representing these concepts as single personified objects. He said, "well, let us begin with the first then, implication, and then you draw an arrow, this is an implication arrow", and he drew =>. Later on, he introduced the symbol for equivalence by saying "if we then take equivalence, then it also is an arrow, but here we have an arrow that is bidirectional" and he drew <=>

Another teacher spoke about two triangles that were enclosed by straight lines, and he referred to them as single alienated objects; "those areas here" and writes "A_1" and "A_2".

Yet another teacher discussed complex numbers and said "you have called the complex numbers z, and z is equal to" and he writes "z = a + bi". The utterance implies that complex numbers were considered objects, and since it was "you" that had called them z, the object was considered to be personified. At the same time, the whole symbolic expression on the right hand side of the equal sign was also referred to as one object, since this was to be considered as a complex number.

Symbolic expressions like the following, with corresponding utterance, are instead considered as alienated objects, because the whole expression was treated as a single object. "$x^2 - 25 = 0$" and "$x^2 - 25x = 0$", "so these are quadratic equations".

6.2.4 Alienated object-talk

A typical pattern for alienated object-talk is in relation to the equal sign, that is, "=" was treated as something static without human interaction. Half of the 92 occasions in this category concerned the equal sign. Furthermore, symbols for numbers were practically always talked about as objects and mostly alienated.

Examples: One teacher showed two expressions, where one was "x = 3" and said "the second one is x is equal to three".

In another class one teacher concluded that "and x is equal to plus or minus the square root of minus one" and wrote "$x = \pm\sqrt{-1}$".

One teacher referred to $\sqrt{2}$ as "this number, square root of two". Later on, the same teacher wrote "$x_1 = 1 + 2i$ and $x_2 = 1 - 2i$" and said "this is an example of a complex number".

> Another teacher, also discussing complex numbers, wrote "Re z = a" and said "the real part of z is equal to a, which is equal to r that is equal to the set of all real numbers, so a can be any real number". He continued, "and the imaginary numbers is the imaginary part of z, which is equal to b" and wrote "Im z = b".

6.3 Several categories for the same symbolic expression (RQ2)

As shown in Table 4, several categories for one expression were used for 49 of the 199 various symbolic expressions (including their constituent parts). These were dominated by situations where the way of speaking belonged to two different categories.

6.3.1 Two categories

It can be noticed that the teachers usually started by using talk belonging to one category and then "moved over" to use talk from another category, and end talking about the symbolic expression using this last "category-talk", instead of moving back and forth between talk belonging to the different categories. For about 60 % of the situations, teachers' different ways of talking around a specific symbolic expression concerned different parts of the expression, that is, talk from one category was used in relation to one part of the expression and talk from another category was used in relation to another part of the expression.

> <u>Example:</u> One teacher was discussing the logarithm and wrote "lg(100 · 1000) = lg (100 000) = 5" while saying "hundred times thousand it is hundred thousand, what is the logarithm for hundred thousand? … that is, what should we take ten to the power of to get the answer hundred thousand? … five yes, just count zeros". The expression was written as the teacher spoke, and in the first part of the utterance, "hundred times thousand it is hundred thousand, what is the logarithm for hundred thousand?", both equal-signs were treated as alienated object-talk since, firstly, a calculation was done without human actor and was referred to as "it is", and secondly, an answer was requested by the words "what is". The second part of the utterance, "that is, what should we take ten to the power of to get the answer hundred thousand? … five yes, just count zeros", treated the logarithm as a personified process since the students were encouraged to think of what they should "take ten to the power of".

For about half of the situations when talk belonging to two different categories were used in relation to one symbolic expression, object-talk was used the entire time (for 19 of the 39 occasions, Table 4). The teachers then started by using personified object-talk almost as often as they started with alienated object-talk (11 vs. 8 times). No particular symbols or situations were found to be characteristic for when both personified and alienated object-talk was used.

Of the 16 occasions when one expression was discussed using both process- and object-talk, 12 occasions consisted of personified process-talk. For these, the object-talk was personified half the times and alienated the other half. It was as common to start by using process-talk as to start by using object-talk. As above, no particular symbols or situations were found to be characteristic for these situations. There were only four of the 39 symbolic expressions for which teachers used both personified and alienated processes-talk, and therefore, no conclusions could be made from these.

6.3.2 Three and four categories

As seen in Table 4, using three different categories of talk in relation to one expression occurred eight times. Of these, seven included personified process-talk together with the two categories of object-talk, and one occasion included alienated process-talk together with the two categories of object-talk. Both personified and alienated process-talk were not used simultaneously when teachers' utterances, in relation to one expression, belonged to three different categories.

Only two teachers used all four categories to speak about a specific expression, one time each. In both cases, the teachers moved back and forth between the categories. Specific for these situations seemed to be that the teachers really wanted to motivate or explain why the particular expression "looked the way it did".

> Example: The teacher motivated the equality "$a \cdot b = 10^{\lg a} \cdot 10^{\lg b}$" by the following utterances; "the number a then, that you can if you want write as ten to the power of lg a". In this instance, utterance in relation to $10^{\lg a}$ was interpreted as *personified object-talk* since you could decide if you wanted to write a in this way. The first utterance was followed by "lg a that was the number that ten should be raised to in order to give the number a", which was interpreted to treat the logarithm as an *alienated process* since the students were encourage to think about the logarithm in this way. The teacher continued, "then we take ten to the power of the number that ten should be raised to in order to give the number ten", which thus was interpreted to treat the logarithm as a *personified process* since now "we" were doing the process. This was followed by "which in turn implies that this then is the same thing as a", which was interpreted as treating the equal sign as a static *alienated object*, an equivalence between a and $10^{\lg a}$. The teacher then moved on to talk about the rest of the expression and said "in the same way we can write the number b, must then be ten to the power of lg b". Since it was "we" who decided how to write the number, the utterance in relation to $10^{\lg b}$ was considered *personified object-talk*. Finally, the teacher concluded "so a is the same thing as ten to the power of lg a and b is the same thing as ten to the power of lg b", which was interpreted to treat the numbers and the equal sign as *alienated objects*.

7 Discussion and conclusions

In relation to RQ1, our results show that the discourse around symbols is slightly more personified than alienated, but clearly more object-oriented than process-oriented, among the teachers from upper secondary school (see section 6.1). This indicates that there is a high degree of reification in the discourse, but not of alienation. Our results provide more empirical data about characteristics of mathematics discourses. This could, among other, give meaning to the common claim, discussed in the background, that mathematical language is compact (cf. Österholm & Bergqvist, 2013). In particular, since one part of objectification can be to condense descriptions of processes and actions into a single noun or short noun phrase, our results shed light on how compact or dense (with respect to objectification) verbal mathematical discourses are and how the compactness vary in mathematical discourses. This is one part required to understand the relation between teachers' teaching and students' learning (Sfard, 2016). Our results from RQ1 can be used to compare with previous, and future, analyses of other types of discourse, to reach a better characterization of variations in mathematical discourse. Below we compare our results with previous results and also address the need of certain types of future studies.

Although using a slightly different method, our results are in accordance with Güçler (2013), where 82 % of the university lecturer's utterances treated "limit" as an object. Corresponding proportion in our study is 76 % (see Table 3, 199 out of 260) for upper secondary teachers' utterances about various mathematical content. Similar to Güçler (2013), Sfard (2016) focuses only on the reification aspect of the objectification when analysing a grade-11 teacher's discourse about quadratic inequalities. Contrary to both ours and Güçler's results, this teacher's discourse (Sfard, 2016) lacked almost any talk about abstract mathematical objects, and was therefore primarily process-oriented. However, there was a lot of talk about concrete mathematical objects (mainly algebraic symbols), all in relation to the students' manipulations of these symbols (Sfard, 2016), which implicitly indicates an absence of alienation in the discourse. By analysing degrees of both reification and alienation, our results provide a more nuanced picture of the objectification in mathematics discourses. Our results imply that teachers primarily objectify through reification, since this dominates regardless of whether the discourse is personified or alienated. At the same time, the discourse is slightly more personified than alienated, particularly when teachers use process-talk in relation to the symbols. Thus, this connection between process-talk and personified discourse resembles the result in Sfard (2016).

From our study's ontological standpoint, teachers need to model the mathematical discourse they want students to develop (Sfard, 2016). By being immersed in a new type of discourse, students develop new ways of speaking of mathematical concepts through reification and alienation, which is substantial in their learning process (Sfard, 2008). Thus, it is reasonable that the teachers' discourse is more advanced than the students', and more objectified, through for example reification, than processual, as our results indicate. At the same time, the degree of objectification would be dependent on how familiar the mathematical content is to the students. A highly objectified discourse by teachers can put high demands on students when interpreting and trying to understand their teacher, since it might be needed that the students have reached a certain degree of objectification in their own thinking in order to be able to participate in a more objectified discourse (cf. Caspi & Sfard, 2012). As shown by both Österholm (2012) and Güçler (2013), university students' mathematical discourse tends to be more process- than object-oriented. There was a notable discrepancy between the objectification degrees of the instructor's and the students' discourse, 82 % compare to 42 % (Güçler, 2013). Both Güçler (2013) and Sfard (2016) are case studies, therefore it is valuable to examine progression in teachers' verbal discourse in relation to students' discourse in upcoming studies in order to be able to generalise any patterns.

In relation to RQ2, we see that our results of the characterization of the categorisation seem to include different situations in relation to how familiar the symbol is for the students. This connection to familiarity of symbols to students is based on our previous overarching knowledge of the Swedish school system and curricula (cf. Swedish National Agency for Education, 2017, 2018) and is therefore discussed here as an interpretation and for potential implications of our empirical results. We here discuss three main types of situations.

First, the results show that when using object-talk, teachers usually address the equal sign and numeric and algebraic symbols in equations (see sections 6.2.3 and 6.2.4). These types of symbols are very familiar to the students at upper secondary level, since they have encountered the equal sign and numeric symbols since the beginning of school and algebraic symbols in equations several years in school. The objectification of these types of symbols is therefore very reasonable at the upper secondary level, when students have had several years of experience of algebra, and the students should have come quite far in their own algebraic discourse, where "algebraic expressions count as signifying fully fledged objects" (Caspi & Sfard, 2012, p. 51). However, the mere magnitude of exposure to algebra over several years does not guarantee that the students' discourses have developed in a qualitative manner, concerning objectification.

Second, the results show that when using process-talk, teachers usually address the handling of expressions and the solving of equations (see sections 6.2.1 and 6.2.2). In these situations, a very familiar concept and phenomenon is active; equivalence, but for which a symbol is seldom used by the teachers. This result, in particular in combination with the result in the first type of situation above, highlights the importance of symbols in objectification, which has also been stressed by other researchers (e.g., Caspi & Sfard, 2012; Tall et al., 2001). That is, despite handling a very familiar type of concept, the discourse is not very objectified when no symbol has been introduced for this concept.

Third, when one teacher introduces new symbols, for implication and equivalence, he mainly uses personified object-talk (see section 6.2.3). A reasonable interpretation is that the objectification process is started by discursively turning processes into objects through these acts of reification (Sfard, 2008). Our results show that the discourse in these situations is more objectified than in the situations where no symbol has been introduced for the same concepts. This strengthens the empirical basis for the connection between the use of symbols and degree of objectification in discourse (cf. Caspi & Sfard, 2012).

Based on these three types of situations, we see a variation in objectification based on content, which is reasonable from a theoretical perspective since students should develop a more objectified discourse when developing their knowledge about a specific content. That is, discourse around more familiar content should be more objectified, which is also the case in our data. This type of variation in teacher discourse could imply that the teachers adapt their mathematical discourse based on students' experiences and prior knowledge, which could be seen as part of their pedagogical content knowledge (Hill, Ball & Schilling, 2008). As described in the background (section 2.1), teachers can support students' development of mathematical discourse in two ways, either by modelling the discourse they want students to adopt or by more explicitly encouraging students to use the desired discourse (Sfard, 2016). Therefore, as part of their pedagogical content knowledge, teachers need to be aware of the possibilities of variation in discourse we have found, and to be able to plan for students' progression of objectification in the mathematical discourse. Another empirical study highlights this important aspect of teaching since "teachers' lack of attention to their own and their students' discourses may contribute to communicational failures in the classroom" (Emre-Akdoğan et al., 2018, p. 1617). However, our data does not allow us to conclude if the variation in teacher discourse that we see reflects more conscious choices or strategies among the teachers, or if this pattern in the discourse reflects more implicit knowledge. It could also be that our data shows the teachers' discourses in a more "static" manner, that is, that the variation in discourse over content reflects their present level of knowledge about different content. These various possible reasons could in turn be related to their years of experience as teachers. Other types of studies are needed to explore these different possibilities, for example, through interviews with teachers and observations of teachers' planning processes, which can give complementing information about how the discourse varies. In particular, more information is needed about the variation in discourse for individual teachers, depending on situation and over time. Such information would be useful for understanding important issues in teacher practice concerning if and how different aspects of teacher knowledge are in use when we notice variations in teacher discourse. We have not found any study focusing on the objectification in teacher discourse over time, but our results about shifts between different degrees of objectification give some information about variation over time, at least at micro level, as discussed below.

In relation to RQ2, our results show that it is quite unusual that the teachers in one and the same situation (for the same symbolic expression) shift between different degrees of objectification (see section 6.3). When this does happen (see section 6.3.2), it has been in different types of situations, and the analyses have not revealed any clear pattern. In the two cases when teachers shift between all four categories of objectification, the teachers seem to try different means of explaining, that is, to use variation in order to reach (all) the students. Thus, in these situations, it seems that the teachers in our study do not consciously model changes in the discourse, but instead change their discourse based on events in the moment during teaching. That is, they do seem to adapt the discourse in ways that they think are most effective (concerning students' learning). It would be relevant to explore this phenomenon in more in-depth analyses in future studies. Somewhat similar result is found by Güçler (2013), where the university teacher only in a few occasions seemed to change the degree of objectification in his discourse that could be regarded as some kind of conscious modelling. However, the changes were not made explicit to the students, which Güçler (2013) argues could be one reason for students' lack of understanding of the specific mathematical content that had been in focus at those occasions. In another study, Güçler (2016) shows that if the metalevel rules of the discourse are made explicit to the students, this could instead support students' understanding.

As noticed by other researchers, empirical analyses of relations between teachers' discourse and students' learning at this detailed level are rare (e.g., Güçler, 2016; Hiebert & Grouws, 2007; Sfard, 2016). When studying objectification, empirical data have mostly been used to examine more general similarities and differences between teachers' and students' mathematics discourses (see the background). To approach a deeper understanding about relations between characteristics and variations in teachers' discourses on the one hand, and students' learning on the other hand, there is a great need for more empirical studies, particularly concerning objectification. One important prerequisite for such studies is more in-depth knowledge about what aspects and degrees of objectification there are in various situations and for different kinds of mathematical symbols in teachers' discourses, which is our contribution by this study.

Our analyses have focused on a random selection of teachers and on their discourse in relation to their use of different types of mathematical symbols in the context of the observations. Controlling the selection of teachers, the selection of situations, and/or the selection of mathematical symbols, and also by focusing on variations for single teachers, would result in other relevant types of analyses, complementing the analyses performed in our study. Although we have used a random selection of teachers, we argue that our results cannot be generalized very broadly. Except as discussed above, we see the need for similar studies for other types of teachers and for other school levels. For example, comparative analyses of different school levels could give valuable information, at macro level, if and how there is a connection between familiarity with mathematical symbols and degree of objectification. We also see a need for other types of selections than random, to be able to focus on more specific issues, including such issues that have been highlighted in our analyses. For example, more in-depth analyses of situations when a new symbol is introduced could give valuable information, at micro level, about how variations in degree of objectification can exist and function in teaching situations.

Besides these needs of different types of studies, our results also highlight specific phenomena that need to be analysed further in future studies. Several examples have been described above, including situations when teachers shift between different degrees of objectification. For example, in many situations, teachers' discourse had a certain type or degree of objectification for one part of an expression but another type or degree of objectification for another part of the same expression. These types of situations could be analysed more in-depth in future studies, focusing on qualitative and explanatory aspects where a broader context is considered, to understand the dynamic of using different types or degrees of objectification in mathematics discourse. Thus, our exploratory research design has fulfilled its purpose by revealing specific phenomena that need to be analysed further in future studies. By not focusing purely on qualitative issues nor purely on quantitative issues, there are certain limitations of this study. In particular, the results are not generalizable (as could be the case from a purely quantitative study) and the results do not include a broader contextualization of data (as could be the case from a purely qualitative study). However, by combining quantitative analyses with qualitative analyses, the study has revealed the existence of certain phenomena that could be relevant to focus on in future studies, which could be either more purely quantitative or more purely qualitative.

In summary, in this study we have characterized discourse of mathematics teaching at upper secondary level around the use of mathematical symbols, concerning variations of objectification. In our analyses, we have seen that the mathematical discourse is highly reified at this level of schooling, compared to being processual, but not more alienated than personified. We make no claims that the mathematics discourse at upper secondary *should* be more, or less, objectified than for example discourses for other subjects at upper secondary level or for other school levels. This kind of comparison is relevant for studies focusing on the overall mathematical Discourse (cf. Gee, 1996) or more generic educational discourse studies (e.g., Herbel-Eisenmann, Wagner & Cortes, 2010). Furthermore, we could also see in the analyses that there exist patterns in the variation of the degree of objectification, in particular that the discourse is more objectified when more familiar symbols are used. Thereby, we have contributed with knowledge about properties of mathematical discourse, not directly connected to a specific mathematical concept or symbol, which is important as a basis for studying "the more general issue of the acquisition of mathematical ways of speaking or writing that may be applicable and acceptable in a wide range of areas of mathematics", which is missing in mathematics education research (Morgan et al., 2014, p. 851). Through our focus on objectification, which is a central idea in many theories of mathematical learning, we have contributed with more general perspectives on mathematical ways of speaking, which are applicable in a wide range of areas in mathematics and in the search for if and how certain properties of classroom discourse can help students develop their own mathematical discourse.

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