Research in Mathematics Education

Routledge
Taylor & Francis Group

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/rrme20

Exploring students' conceptual understanding through mathematical problem solving: students' use of and shift between different representations of rational numbers

Jonas Jäder & Helena Johansson

To cite this article: Jonas Jäder & Helena Johansson (04 Feb 2025): Exploring students' conceptual understanding through mathematical problem solving: students' use of and shift between different representations of rational numbers, Research in Mathematics Education, DOI: 10.1080/14794802.2025.2456840

To link to this article: https://doi.org/10.1080/14794802.2025.2456840

Published online: 04 Feb 2025.

RESEARCH IN MATHEMATICS EDUCATION
https://doi.org/10.1080/14794802.2025.2456840

Routledge
Taylor & Francis Group

OPEN ACCESS Check for updates

Exploring students' conceptual understanding through mathematical problem solving: students' use of and shift between different representations of rational numbers

Jonas Jäder and Helena Johansson

Department of Engineering, Mathematics and Science Education, Mid Sweden University, Sundsvall, Sweden

ABSTRACT
One way to stimulate conceptual considerations is through mathematical problem solving, which requires students to construct new solution methods, potentially including new representations. Hence, this study focuses on students' use of and shift between representations during problem solving, specifically regarding the fundamental and often challenging concept of rational numbers. Swedish elementary school students were observed when tackling mathematical problems in pairs. The findings revealed that the students' considerations concerned several conceptual properties of rational numbers through the use of different representations. However, students tended to use the representations presented in the problem, favouring the most familiar representation rather than constructing their own. As a result, they may have overlooked opportunities to explore a wider range of representations which could have deepened their understanding of the concept. This suggests a potential opportunity to design problems that require students to transfer between representations and grapple with unfamiliar ones.

ARTICLE HISTORY
Received 23 May 2024
Accepted 10 January 2025

KEYWORDS
Conceptual understanding; problem solving; representations

Introduction

Doing mathematics is based on an understanding of relevant mathematical concepts (Stein & Smith, 1998). Conceptual understanding constitutes a crucial element of mathematical knowledge (Hiebert & Grouws, 2007; Kilpatrick et al., 2001), as evidenced in numerous national curricula, including that of Sweden (Skolverket, 2022). However, what is meant by understanding a concept is frequently vaguely defined, often focusing more on identifying what students can do rather than anything else (Simon, 2017). It is thus of importance to link students' actions to specific aspects of conceptual understanding. One central aspect of mathematical concepts and of conceptual understanding is the interaction with and use of representations (Viseu et al., 2021; Wedman, 2020). The

CONTACT Jonas Jäder jonas.jader@miun.se; jonasjader71@gmail.com Department of Engineering, Mathematics and Science Education, Mid Sweden University, Sundsvall, Sweden

ability to transfer between different representations is considered essential for the development of conceptual understanding (Duval, 2006). Explanatory models can be seen as didactical tools making it possible to explore particular properties of a concept by using specific representations, both concrete and abstract. These models can aid learning by providing different perspectives and support (Van den Heuvel-Panhuizen, 2003). Conceptual understanding needs to be developed through personal experiences rather than by replicating a formal definition with little conceptual awareness or conceptual consideration (Niss, 2006). By including several diverse learning situations, a student is provided with the opportunity to explore and understand different properties of a concept.

The struggles and conceptual challenges inherent in a mathematical problem-solving process are avenues through which persistence and conceptual understanding can be cultivated (Hiebert & Grouws, 2007; Jäder, 2022; Sullivan et al., 2015). This highlights the importance of understanding underlying principles rather than simply memorising or imitating procedures (Lithner, 2017). Mathematical problem solving is generally defined to include the construction of new solution methods (Schoenfeld, 1985) and has proven to be valuable for mathematical learning (Jonsson et al., 2014). When used thoughtfully, uncertainties and challenges can be productive elements in students' learning processes (e.g. Kapur, 2014; Zaslavsky, 2005). Additionally, it is of importance that the design and analysis of mathematical problems have a clear, potential purpose or learning goal in mind (Jones & Pepin, 2016; Pepin, 2012).

Mathematical problems typically require construction of new representations (Lester, 2013), which as previously highlighted, is an important aspect of conceptual understanding. Additionally, it has been demonstrated that students' conceptual challenges in mathematical problem solving can be identified and characterized, such as navigating the connections between different representations and meeting unfamiliar situations (Jäder, 2022). Mathematical problems and specific features of these problems may also be valuable resources, aiding not only in the development of conceptual understanding but also in the exploration of students' conceptual understanding (Mitchell & Clarke, 2004). Taking into account the need to differentiate between the ability to perform a task and the (conceptual) understanding revealed through this action, as well as recognising that different tasks may stimulate different conceptual considerations, the aim of this study is to contribute to the field's understanding of mathematical problem solving as a means to explore students' conceptual understanding.

A mathematical concept which has proven to be challenging and at the same time is considered to be a threshold concept for students is that of rational numbers (Siegler et al., 2012; Stafylidou & Vosniadou, 2004). Therefore, this study focuses on young students' use of and shift between different representations in relation to the concept of rational numbers within the context of mathematical problem solving. The research question addressed by the study is:

- What conceptual considerations regarding rational numbers are made visible in students' use of and shift between different representations during mathematical problem solving?

Mathematical concepts and conceptual understanding

Although mathematical concepts are considered central in mathematics education, there is no unanimous definition of the notion. Nevertheless, a mathematical concept typically refers to an abstract idea (Wedman, 2020), such as a formal theoretical definition (Sfard, 1991; Tall & Vinner, 1981; Vinner, 2020), or an individual construction (Watson & Mason, 2006). Simon (2017) elaborates on the notion and defines a mathematical concept as "a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (p. 123). Through systematic literature reviews of philosophical discussions concerning concepts and mathematics education research related to mathematical concepts, Wedman (2020) identified five aspects that can be used to describe a mathematical concept: *definition and properties*; *typical and atypical objects*; *hierarchical and non-hierarchical relations*; *processes and procedures*; *representations*. All five aspects are considered essential for fostering conceptual understanding in mathematics (Wedman, 2020).

The first aspect, definition and properties, refers to the idea that mathematical definitions can be considered distinct categories, each with its own unique set of properties (ascribed attributes) that characterise a specific concept (e.g. Harel & Weber, 2020). Conceptual understanding involves the identification and processing of relevant (intrinsic) properties, while distinguishing them from superficial ones (e.g. Lithner, 2008; Wedman, 2020). Typical and atypical objects is a second aspect which relates to the idea of prototypes as typical examples of different categories of mathematical objects (Presmeg, 1992). Diverse experiences can result in variations in the prototypes of mathematical objects. For instance, in some classrooms the diameter of a circle is consistently depicted horizontally, while in others it is shown vertically. Research has shown that both teachers and students often rely on typical and representative examples of mathematical objects when engaging in activities such as testing properties for a particular concept (Alcock & Simpson, 2002; Presmeg, 1992). Therefore, the selection and utilisation of prototypes significantly influence students' conceptual development processes. Thirdly, conceptual understanding is seen as encompassing an understanding of both hierarchical and non-hierarchical connections between mathematical concepts (Murphy, 2004). The distinction between hierarchical and non-hierarchical relations among mathematical concepts stems from the idea that relations can be described as either hierarchical tree-structures or non-hierarchical web-structures. This applies both within the same mathematical content area and between concepts from different content areas (Murphy, 2004; Wedman, 2020). The fourth aspect relates to the idea that a mathematical concept cannot be separated from how it is used (Sfard, 2008). Initially, the focus may be on employing procedures to manipulate a symbolic expression in one way or the other. Over time, this may evolve into viewing a process or procedure as a mathematical object rather than an automated step-by-step scheme (Gray & Tall, 1994; Sfard, 2008).

The fifth aspect, representations, is widely regarded as central to mathematical activities and relates to the various ways in which mathematics is communicated, both with oneself and with others (e.g. Duval, 2006; O'Halloran, 2005; Schleppegrell, 2007; Sfard, 2008). The term "representation" carries different meanings across different theoretical perspectives. Additionally, alternative terms such as "modes", "semiotic systems", "signifiers" or "visual mediators" are sometimes used to refer to resources for communication (e.g. Duval, 2006; O'Halloran, 2005; Sfard, 2008). Examples of different mathematical representations include mathematical words, numbers, mathematical symbols, diagrams, pictures and physical artifacts. Different representations offer the opportunity to explore different perspectives of the same mathematical concept, and a particular concept can be easier to understand through one representation compared to another (Usiskin, 2018). An important aspect of conceptual understanding involves grasping how different representations of a particular mathematical concept are related to each other. This includes the ability to transfer between different representations for the same concept (Duval, 2006; Wedman, 2020).

The concept of rational numbers and students' understanding of rational numbers

The concept of rational numbers is an important area of mathematics education (e.g. Confrey et al., 2009). Research has demonstrated that an understanding of rational numbers predicts success in higher mathematics in general and in algebra in particular (Clarke & Roche, 2009; Siegler et al., 2012; Torbeyns et al., 2015). Moreover, rational numbers are a key concept when developing algebraic thinking and working algebraically (Eriksson & Sumpter, 2021; Loewenberg Ball et al., 2005).

Understanding the concept of rational numbers implies being able to consider a rational number as a part-whole (fraction), as a decimal, as a ratio, as an indicated division (a quotient), as an operator (multiplicative) and as a measure of continuous or discrete quantities (numbers with a given magnitude) (e.g. Behr et al., 1983; Confrey et al., 2009; Kieren, 1976; Vamvakoussi, 2015). In some literature, these distinct interpretations are discussed as categories of rational numbers (Gabriel et al., 2023), as sub-constructs to rational numbers (e.g. Kieren, 1995; Wright, 2014), or as different ways of representing rational numbers (González-Forte et al., 2023). In this study we will use sub-constructs when referring to the different ways of understanding rational numbers. Moreover, the term "fraction" is not consistently used in the literature as a delimited sub-construct of rational numbers. Instead, it is sometimes used more broadly and interchangeably with rational numbers (e.g. Lee & Lee, 2023; Pedersen & Bjerre, 2021). This study follows Olive (1999) and consider fractions as one sub-construct of rational numbers.

Research indicates that students' difficulties with developing an understanding of rational numbers partly because they attempt to apply properties of whole numbers to them, often referred to as the Natural Number bias (NNB) (e.g. Gabriel et al., 2023; González-Forte et al., 2020; Stafylidou & Vosniadou, 2004; Van Dooren et al., 2015). This includes, for instance, the tendency to assess the magnitude of a rational number by considering the sizes of the numerator and denominator independently (González-Forte et al., 2023; Stafylidou & Vosniadou, 2004). It also involves treating rational numbers as discrete entities, where there is always a fixed number before or after a particular number, despite the fact that there are infinitely many numbers between any two rational numbers (Gabriel et al., 2023). Properties of natural numbers are often based on additive thinking, whereas for rational numbers it is essential to depart from multiplicative thinking (Van Dooren et al., 2015). This change from additive to multiplicative thinking has been shown to be demanding for students (e.g. Moss, 2005). Other difficulties arise with equipartitioning (Confrey et al., 2009; Mack, 2001) particularly when representing equipartitioning pictorially. Interestingly, this form of representation seems to be most familiar to younger students when introduced to rational numbers (Viseu et al., 2021). Both equipartitioning and the development of multiplicative thinking are regarded as foundational for understanding rational numbers (Confrey et al., 2009; Simon & Tzur, 2004; Wright, 2014). It is also suggested that fraction is the most difficult sub-construct of rational number to understand. Conceptual understanding of fractions involves recognising their infinitely varied equivalent forms as different representations (e.g. 1/2, 2/4, 5/10, 1234/2468, etc) (Gabriel et al., 2023).

To support students' conceptual development with regard to sub-constructs of rational numbers, different pedagogical representations or explanatory models are often used (Lee & Lee, 2023; Norton & Wilkins, 2009; Van den Heuvel-Panhuizen, 2003; Vig et al., 2014). The three most common models are the *area*, *set* and *length* models. In the area model, geometric shapes are subdivided into equal parts. The set model uses a set of countable objects subdivided into equal shares. Finally, the length model uses fractions strips, bars or the number line, all subdivided into a designated unit (Lee & Lee, 2023; Van den Heuvel-Panhuizen, 2003). As explanatory models possibly only refer to specific properties of a concept rather than the full abstract idea of the concept, their use may create confusion and even contradictions and cognitive conflicts (Ahl & Helenius, 2021; Vig et al., 2014). It has also been shown to be important to use tasks that will help students overcome NNB. For instance, fraction comparison tasks that cannot be solved using knowledge of natural numbers have been found to be helpful (e.g. comparing 2/3 and 4/9 instead of 5/8 and 3/6) (Gabriel et al., 2023 González-Forte et al., 2023). Similarly, tasks that require multiplicative rather than additive thinking have been shown to be beneficial (Van Dooren et al., 2015).

In the present study, primary focus is on students' use of and shift between different representations. The mapping of what conceptual understanding is, specifically concerning rational numbers, has served as the foundation for selecting and designing tasks. However, these distinctions will not be further detailed in the analysis or results.

Method

In order to explore students' conceptual considerations during problem solving, 24 students from grades 2, 5 and 6 were asked to solve tasks expected to be mathematical problems to them. The problem-solving process was analysed with a focus on students' use of and shift between representations in relation to specific sub-constructs of rational numbers.

Selection of students

The study was conducted in collaboration with a primary school in a small Swedish town with which there was an established contact, and involved teachers in grades 2, 5 and 6. The school is regarded as an ordinary Swedish primary school where most students live nearby in a catchment area including neighbourhoods of varied socioeconomic status, however not including any extremes. Students in these three classes (and their legal guardians) were asked if they were willing to solve some mathematical problems in pairs and to be filmed while doing this. As most students were willing to participate, the three class teachers were asked to select eight students each to form couples that were likely to be able to cooperate and to share their thoughts with each other. There were no other criteria for selection, and the teachers were for example clear about have selected students with a varied level of mathematical competence.

Ethics

The research follows Good Research Practice (Swedish Research Council, 2017). Informed consent was accordingly collected in written form from the students' and their legal guardians. In compliance with the Act concerning ethical review of research in Sweden (The Ethics Review Act SFS 2003:460, 2003), the research has not undergone ethical review since this is not required for the type of data collected in this study.

Selection and design of mathematical problems

In line with the purpose of the study, the objective of the selection and design process was to identify tasks that would present reasonably challenging mathematical problems for the students. While the primary goal for the students may be to solve the problem, the design of mathematical problems should also take into consideration that the activities provided should also offer opportunities for learning (Ainley et al., 2006). Four main principles guided the selection and design of mathematical problems used in the study.

- The solution method, consisting of a number of steps required to be performed to solve the task, included one or several (not necessarily major) steps expected to be new to the students.
- The level of challenge needed to be both reasonable and varied, ensuring that students with different levels of competence could all meet a challenge in some of the problems.
- The problems would represent various explanatory models related to rational numbers.
- The problems should generate a need to shift between different representations of the concept rational numbers in the solution process.

The design process was based on these principles, as well as previous research and the researchers' experience with students' understanding of rational numbers. There were two parallel design processes, one for grades 5 and 6 and another for grade 2 students.

Problem-solving sessions

The students' teachers suggested pairs of students to work together at the school, ensuring that each pair would be made up of students likely to cooperate during the problem-solving activities. The first author of the paper attended each session. The problem-solving sessions were conducted in a small room adjacent to the classroom. The student pairs were presented with one problem at a time, on a paper, initially with no additional information, and were asked to think aloud (Schoenfeld, 1985). During the students' problem-solving process, the primary role of the researcher was to make sure that each of the problems was correctly understood by the students. When necessary, the researcher would ask the students about their thoughts and seek clarification of their reasoning. To potentially gain further insights into students' work with mathematical problems and their conceptual understanding, the decision was taken to also use enabling prompts.

Enabling prompts (Sullivan et al., 2015) and well-prepared questions, both general and task-specific (Olsson & Granberg, 2024), have been shown to facilitate the implementation of challenging tasks. In this study, the focus was on students' understanding and use of representations, especially when it was obvious that the students would not reach a final, correct solution. This was done in two main ways: through simplifying or clarifying the problem, or through challenging an incorrect conclusion drawn by the students. These modes of interaction could also be linked to the inclusion or alteration of representations.

Procedure for analysis

Initially, the solution process of each pair of students for each of the tasks was categorised as mathematical problem solving or routine work. For the process to be considered mathematical problem solving it was deemed necessary for there to be some indication, either through the students' actions or their statements, that (part of) the solution method was new to them compared to prior experiences. All instances where students were involved in mathematical problem solving were then further analysed. Consequently, only those instances that appeared to be mathematical problems from the students' perspective were included in the final analysis.

Secondly, using the concept of conceptual challenge (Jäder, 2022), the next step of the analysis procedure entailed indicating any instance in the solution process where a student showed signs of hesitation, struggle or resistance related to the mathematical concepts relevant for the task. These instances were further investigated as described below.

Thirdly, students' use of and shift between representations of rational numbers were identified from the general description of their solution process. It was also noted how students interacted with these representations. This included whether the representation was provided by the problem formulation, or if it was a result of an interaction with the researcher such as simplification, clarifications or challenging (as previously mentioned). Additionally, it was noted if the representation was a result of the students' own initiative, either as a trigger response to the problem, or as a means to reach a conclusion. Furthermore, particular attention was paid to any transitions made between different representations.

Linked to each instance described in the third step, the properties of rational numbers that the students were considering were described. In addition, other conceptual aspects such as atypical objects or relations to other concepts were used to describe students' work.

Summary of the analysis procedure:

- Routine task or mathematical problem
- Indications of conceptual challenges
- Characterisation of students' use of representations

- Researchers' interaction with students (none, simplify/clarify or challenge)
- Link to properties of the concept of rational numbers and to other aspects

Results

All but one of the tasks used for each of the grades were considered to be relevant mathematical problems. However, one of the tasks used in grade 6 proved to be too challenging and was consequently excluded from the sessions performed later with grade 5-students. Four pairs of grade 2 students' work with seven mathematical problems was analysed, while for grades 5 and 6, a total of eight student pairs' work with seven mathematical problems was analysed.

In summary, the results show that students in grade 2 as well as grades 5 and 6 demonstrate similar approaches to the shift between representations. In none of the problems did any of the students construct their own representations; instead they relied solely on the representation presented in the problem, such as symbols, texts, or pictures. On some occasions it was evident that students encountered an unfamiliar situation or representation in relation to the concept of rational numbers. For example, certain geometric figures illustrating rational numbers using the area model seemed to have been unfamiliar to the students. Similarly, the number line seemed unfamiliar to the students in relation to the concept of rational numbers. Moreover, it appeared that students were not able to commit to more than one explanatory model at a time, unless explicitly required to do so.

The students were observed to take their point of departure from what was well-known to them in the representations. In most cases, some kind of clarification or a simplification of the problem was necessary for the students as they were not able to reach a conclusion. In yet other instances, students reached a conclusion, albeit incorrect. In these instances, as outlined in the method, the researcher intervened during the problem-solving sessions, challenging the students' conclusions. The results below are structured according to these three types of interactions: none, clarification/simplification and challenging.

No interaction

Grade 2 students to a higher degree were able to reach a correct conclusion on their own. This applied to almost half of the problems. To a lesser degree, grade 5 and 6 students arrived at a correct conclusion without any interaction with the researcher.

Figure 1 shows an example of a problem that grade 2 students were able to solve on their own.

The task emphasises the sub-constructs part-whole relationships and rational numbers as a quantity, as well as the relation between different representations for a particular rational number. The challenges noted in this task concerned transferring between symbolic and different iconic representations regarding one third (subtask f). It was observed that students encountered a challenge using the number line as a representation. They often ignored the 0 and the 1 as representing numbers (i.e. considering the number line as a length model) and instead used these markings as start and end points of an interval that should be divided into equal parts, i.e. part of whole. To deal

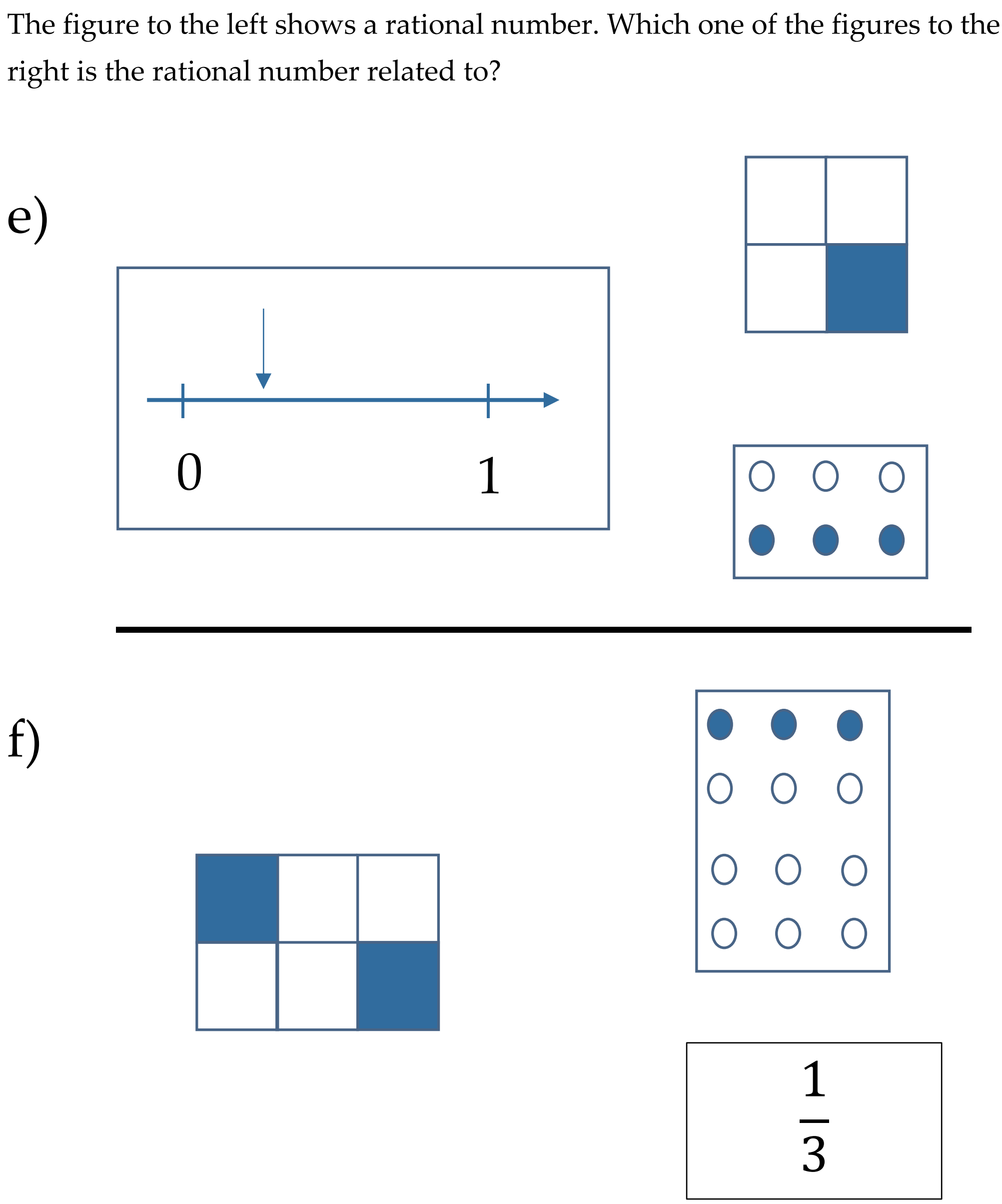

Figure 1. Example of a problem solved by grade 2-students without support.

with the challenge, students mostly used a strategy of exclusion to come up with a solution. For example, in subtask (e) they concluded that the bottom right figure represented one half, and then that the arrow did not divide the interval into two equal parts. As a result, they concluded that the relation was between the number line and the upper right square. This conclusion was drawn without considering the specific rational numbers represented by the number line and the square (i.e. they never mentioned one-fourth). Students in grades 5 and 6 solved similar problems appropriate for their age in a comparable manner, without any support aside from clarification of the task itself rather than specific mathematical content. The students needed to be able to interpret rational numbers in relation to different explanatory models as well as different representations such as iconic, linear, symbolic and text. One area of focus was on rational numbers as a quantity, where the relationship between numerator and denominator makes up an indicated division (a quotient). However, students seemed to mainly regard rational numbers through either the area or the set model. Even though it might be possible that some students regarded the number line as another way of representing a part-whole relationship, they also needed to connect this to the set model.

Clarification/Simplification

There were several examples of both younger and older students needing clarification or simplification to move on. A first example of this is the problem attempted by the students in grades 5 and 6, which is presented in Figure 2.

When working on this problem, students needed to consider the rational number as a numerical value, where the relationship between numerator and denominator determines its magnitude. It appeared that students had a hard time taking a step away from the often-used explanatory model of part-whole, where in this case the number line could represent the whole. For example, one pair of students discussed were to place 4/8 and focused only on the numerator in relation to the reference numbers (0, 1) on the number line, never progressing in the solution process. The researcher then prompted the question "Approximately how much is four eighths?" The students then explained "You have eight squares and colour four of them … can it then be in the middle … you can think of the whole line as eight squares". Possibly, students were unfamiliar with referring to rational numbers in the context, necessitating clarifications.

For grade 2, an example of when simplification was needed can be seen in Figure 3. The figure shows two examples of problems where students needed support in the form of simplification to be able to reach a conclusion.

The problem to the left in Figure 3 focused on the set model, specifically the property parts of a number distinguished from parts of an area. All students began by concluding that Aya takes 10 sweets. Then it became challenging. It was obvious that the students were familiar with equipartitioning in relation to the area model but not to the set model. They expressed that "a fourth is when you divide in four equal parts".

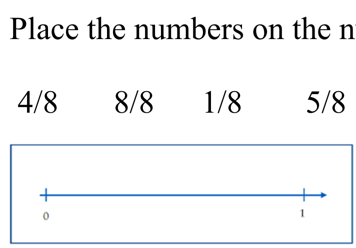

Figure 2. Example of a problem where students needed clarification to solve the problem.

Aya and Beno have 20 sweets. Aya takes half and Beno takes a fourth. How many of the sweets are left for Jonas?	Ludde and Fia eat a chocolate cake. Ludde eats half of the chocolate cake and Fia eats a fourth. How much have they eaten together?
	

Figure 3. Examples of two problems where students required support in the form of simplification.

However, the researcher needed to give the students a simplifying prompt saying that one fourth corresponds to “separating the candies into four piles and taking one of these piles”, whereafter the students realised that the number of sweets could be divided into four “piles” with an equal number of sweets in each pile. Thus, the students transferred between using the representation parts of an area to parts of a number.

The mathematical question in the problem to the right in Figure 3 resembles the problem to the left. The picture of the chocolate cake could be regarded as representing parts of an area (i.e. the area model). The analysis of the previous problem had revealed that all students were most familiar with this model. However, none of the students used the picture of the chocolate cake as a representation in this manner. Instead, all students counted the bits of the chocolate cake and used this as representing parts of a number (i.e. set model). Furthermore, even though students had provided explanations of a fourth in the previous problem, analysis indicated that all four pairs needed some simplifying support from the researcher in terms of how to understand what *a fourth* meant in this context. Consequently, the property one fourth together with fractional parts of a number became the properties in focus in relation to the sub-construct of quotient. Once students had figured out the meaning of one fourth, the calculations were routine.

Challenging an incorrect conclusion

On some occasions, the students came to an incorrect conclusion which could be challenged by the researcher, allowing the students to further explore the mathematical content and gain insights into the concept of rational numbers. One example of this can be seen in Figure 4, which is a problem presented to the 5th and 6th grade students.

The focus was on the property of rational numbers stipulating that the parts need to be of equal size (i.e. equipartitioning) particularly in relation to the area model. For example, students struggled to convince themselves or their partner that it was ok to “create a new part” in the second figure to see that ¼ was shaded. However, on several occasions the researcher needed to challenge the students further to make the necessary conceptual consideration, rather than leaving the problem with an incomplete or incorrect answer. Several of the pairs selected the figure *d*, leaving out one of the correct

Which five figures shows that 1/4 is shaded

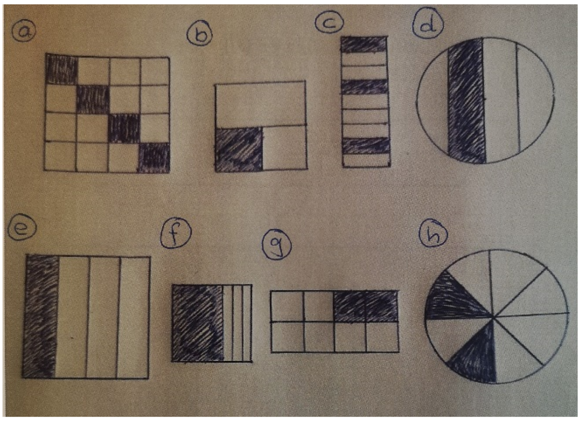

Figure 4. Example of a problems where students were challenged when they initially reached an incorrect conclusion.

choices. Even though students mentioned that it was important that the parts were of equal size, they did not consider this property when selecting the figure *d* as one of the five representing ¼. Additionally, the researcher challenged their answer by, for example, questioning why figure *h* but not figure *a* represents ¼. It was apparent that the way the parts were represented in the figure *h* was more typical for the students. Besides figure *a*, some of the other figures seemed to be less familiar or unfamiliar to the students, such as figure *b*, where the lack of a full vertical line dividing the square made the students hesitate as to how to describe the shaded part, unsure whether it represents 1/3 or something else.

The approach these students took was similar to that of the students in grade 2 when they met a similar problem appropriate for their level of understanding.

Discussion

In line with our objectives, the results of this study show that students' conceptual considerations vary depending on the mathematical problem they encounter. It's unsurprising that different mathematical problems produce different situations and, consequently, different conceptual considerations. This finding is now further supported by empirical data. Considering that a student's use of and shift between representations in relation to a specific concept is an integral part of a student's conceptual understanding (Duval, 2006; Wedman, 2020), it may be reasonable to assume that using different representations can lead to the creation of different opportunities to learn that prompt conceptual considerations and the utilisation of different properties of a concept. Assuming that conceptual considerations enhance students' opportunities to develop an understanding of the mathematical concept in question (Hiebert & Grouws, 2007; Jäder, 2022; Zaslavsky, 2005) we propose that a well-considered design of mathematical problems should incorporate several components. This includes a requirement for students to transfer between various representations, incorporate atypical representations and make use of relevant explanatory models focusing on specific properties of a concept. Moreover, the design process should strive for the inclusion of encouragement for students' conceptual considerations during the problem-solving process. These issues are further discussed below.

As our results show, it appears typical for students to start their problem-solving process by focusing on aspects of the situation that are familiar to them as well as the representations presented in the problem (e.g. symbols or figures). They seldom attempted to use other representations that were not explicitly presented in the problem. This outcome is unsurprising and aligns with previous findings showing that students favour familiar representations and solution methods in task-solving situations (e.g. Alcock & Simpson, 2002; Lithner, 2008; Presmeg, 1992). Knowing that this is an expected behaviour among students, it is essential to account for it when designing mathematical problems. This ensures that students have the opportunity to explore the concept in focus through different representations (cf. Duval, 2006; Niss, 2006), facilitating the need for them to transfer between different representations. This approach acknowledges that each representation may have the potential to highlight certain conceptual aspects, thus enriching students' understanding through multiple perspectives. We argue that one way to do this is to explicitly include more than one type of representation in the problem that requires reflection, such as the example in Figure 1. This type of design also seems to be fruitful for problems that students can work on independently. Conversely, problems that do not highlight different representations may limit students' conceptual development.

This also relates to Wedman's (2020) discussion concerning typical and atypical objects as an aspect of mathematical concepts. In our study we were able to identify familiar representations in unfamiliar situations, as well as unfamiliar representations with respect to rational numbers. For example, the use of the number line as a representation of rational numbers as quantities (Figure 1 and 2), or the use of somewhat unfamiliar figures as representation of rational numbers as part-whole (Figure 4). As can be seen from the results, although students explicitly mentioned the particular property of equipartitioning, they initially were not mindful of this property, especially when dealing with somewhat unfamiliar figures or when required to divide figures into parts themselves. However, the results indicate that as students' conclusions are challenged, the situation where unfamiliar representations are used in relation to the area model seem to trigger students' conceptual considerations and shed light on the notion of parts being of equal size in terms of different geometric figures and on equipartitioning (cf. Viseu et al., 2021). This strengthens the assumptions that it is valuable to consider the use of representations that are atypical in a specific situation, alongside more commonly used ones.

A situation that was evidently unfamiliar to the majority of students, both younger (grade 2) and older (grades 5 and 6), was the use of the number line to represent the length model. Students frequently ignored the 0 and the 1 as numerical representations on the number line instead using these markings as beginning and end points of an interval to be divided into equal parts. In essence, instead of applying the length model, they applied the principals of the area or the set models to the number line. This works as long as focus is on the sub-construct of fractions (i.e. focusing on the part-whole aspect), or when the number line represents values from 0 to 1. However, understanding rational numbers as quantities with given magnitudes requires the use of other explanatory models (cf. Kieren, 1976, 1995). This highlights the importance of not only considering which particular representations are used in the design of mathematical problems but also considering their alignment with explanatory models and consequently how the representation is used (e.g. whether number lines include values above 1 and/or values below 0). A representation, for example corresponding to an explanatory model, may support the visualisation of specific properties of a concept, rather than the entire concept itself (Ahl & Helenius, 2021; Vig et al., 2014). Although it is common to use the area and set models in introductory work on rational numbers, there is a need to broaden this approach to allow students to explore relationships and connections with all sub-constructs of rational numbers. For instance, the area and set models may hamper students from overcoming NNB because most situations involving these models can be seen from an additive perspective (cf. Gabriel et al., 2023; González-Forte et al., 2023). Therefore, there is potential in developing teaching and learning through the use of a linear explanatory model. However, if we want students to experience the linear model, its construction needs to be carefully distinguished from that of the area model.

Moreover, it appears to be beneficial to enhance students' conceptual considerations through additional support, beyond the written instructions provided by the mathematical problem. In this study, two forms of interaction (clarification/simplifications and challenging) were used and demonstrated effectiveness without reducing the mathematical problem to a routine task. Instead, they supported the creation of a problem at an appropriate level of difficulty, presenting reasonable challenges while encouraging conceptual considerations. The absence of interaction between the student and the researcher (or teacher) may indicate that the mathematical problem works well on its own. However, there are instances where students' deliberations are limited to choices rooted in basic ideas rather than fostering conceptual considerations that might develop a deeper understanding of the concept. Therefore, when designing mathematical problems for learning we recommend that the design process includes suggestions for student-teacher interaction depending on students' approach to the problem, akin to the enabling and extending prompts advocated by Sullivan et al. (2015) or questions suggested by Olsson and Granberg (2024). Furthermore, this provides teachers with a greater opportunity to gain insights into students' understanding and progress (Mitchell & Clarke, 2004), which is a crucial aspect of a formative approach to teaching.

Limitations, implications, and conclusions

The findings of this study are based on a convenience sampling of twenty-four students, from the same school. However, given that the selection of students solely considered students possibilities to cooperate with each other, rather than for example level of mathematical competence, understanding of rational numbers or representational skills, and that the focus is on the relationship between students and mathematical problems rather than on specific student behaviours, it is reasonable to assume that the observed relationships can be extrapolated to discuss conceptual considerations in a wider sense, with regard to students and problems in general. Conducting the study in a clinical setting rather than in a classroom may lead to some loss of authenticity, such as how students would react to student-teacher interaction or their attitudes towards approaching these problems. Nonetheless, the setting enabled a necessary and structured set up facilitating a more thorough analysis. The results are clearly dependent on the interaction between researcher and students, which is why we have tried to explain these interactions following specific guidelines and limited to distinct categories (such as request for clarification of thoughts, simplifications of the problems or challenging a conclusion) in a transparent manner.

The results in this study strengthen and expand the research field's knowledge about the importance of including atypical representations, and situations, to foster students' conceptual understanding of rational numbers, as well as the importance of incorporating teacher interaction already in the design and planning of teaching. In addition, our study has strengthened Wedman's (2020) theoretical assumption of five aspects of conceptual understanding, by showing how three of these aspects, *typical and atypical objects, definitions and properties* and *representations*, manifest in empirical settings.

While the study does not specifically focus on students' learning, it may be reasonable to conclude that in order to effectively guide students' learning, teachers need to be aware of their progress and have insights into their conceptual understanding. Considering these limitations, we propose that our findings indicate that it is beneficial for teachers to incorporate multiple and diverse representations, including those that are atypical for the students, as well as representations used in different explanatory models. Further, it seems reasonable for teachers to be prepared to guide students in their problem-solving process by providing additional instructions such as simplifications and challenges tailored to individual students, which may not be possible to include in the original written problem.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by ULF - Utbildning, Lärande, Forskning and Mid Sweden University, Mittuniversitetet [grant number: MIUN 2022/970].

ORCID

Jonas Jäder http://orcid.org/0000-0002-8498-8014
Helena Johansson http://orcid.org/0000-0002-5981-3722

References

Ahl, L. M., & Helenius, O. (2021). Polysemy and the role of representations for progress in concept knowledge. In Y. Liljekvist & J. Häggström (Eds.), *Sustainable mathematics education in a digitalized world. Proceedings of MADIF 12* (pp. 101–110). Swedish Society for Research in Mathematics Education.

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, *32*(1), 23–38.

Alcock, L., & Simpson, A. (2002). Dealing with categories mathematically. *For the Learning of Mathematics*, *22*(2), 28–34.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–125). Academic Press.

Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, *72*(1), 127–138. https://doi.org/10.1007/s10649-009-9198-9

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd conference of the international group for the psychology of Mathematics Education* (Vol. 2, pp. 345–352). PME.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1-2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Eriksson, H., & Sumpter, L. (2021). Algebraic and fractional thinking in collective mathematical reasoning. *Educational Studies in Mathematics*, *108*(3), 473–491. https://doi.org/10.1007/s10649-021-10044-1

Gabriel, F., Van Hoof, J., Gómez, D. M., & Van Dooren, W. (2023). Obstacles in the development of the understanding of fractions. In K. M. Robinson, A. K. Dubé, & D. Kotsopoulos (Eds.), *Mathematical cognition and understanding* (pp. 209–225). Springer.

González-Forte, J. M., Fernández, C., Van Hoof, J., & Van Dooren, W. (2020). Various ways to determine rational number size: An exploration across primary and secondary education. *European Journal of Psychology of Education*, *35*(3), 549–565. https://doi.org/10.1007/s10212-019-00440-w

González-Forte, J. M., Fernández, C., Van Hoof, J., & Van Dooren, W. (2023). Incorrect ways of thinking about the size of fractions. *International Journal of Science and Mathematics Education*, *21*(7), 2005–2025. https://doi.org/10.1007/s10763-022-10338-7

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 115–141.

Harel, G., & Weber, K. (2020). Deductive reasoning in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 183–190). Springer.

Hiebert, J., & Grouws, D. (2007). The effects of classroom mathematics teaching on students learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics* (pp. 371–404). Information Age Pub.

Jäder, J. (2022). Creative and conceptual challenges in mathematical problem solving. *Nordic Studies in Mathematics Education*, *27*(3), 49–68.

Jones, K., & Pepin, B. (2016). Research on mathematics teachers as partners in task design. *Journal of Mathematics Teacher Education*, *19*(2–3), 105–121. https://doi.org/10.1007/s10857-016-9345-z

Jonsson, B., Norqvist, M., Liljekvist, Y., & Lithner, J. (2014). Learning mathematics through algorithmic and creative reasoning. *The Journal of Mathematical Behavior*, *36*, 20–32. https://doi.org/10.1016/j.jmathb.2014.08.003

Kapur, M. (2014). Productive failure in learning math. *Cognitive Science*, *38*(5), 1008–1022. https://doi.org/10.1111/cogs.12107

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Number and measurement* (pp. 101–144). Department of Mathematics Education University of Georgia Athens.

Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31–65). State University of New York Press.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding It Up: Helping children learn mathematics*. National Academy Press.

Lee, J. E., & Lee, M. Y. (2023). How elementary prospective teachers use three fraction models: Their perceptions and difficulties. *Journal of Mathematics Teacher Education*, *26*(4), 455–480. https://doi.org/10.1007/s10857-022-09537-4

Lester, F. K. Jr. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast*, *10*(1), 12.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, *67*(3), 255–276. https://doi.org/10.1007/s10649-007-9104-2
Lithner, J. (2017). Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM Mathematics Education*, *49*(6), 937–949. https://doi.org/10.1007/s11858-017-0867-3
Loewenberg Ball, D., Ferrini-Mundy, J., Kilpatrick, J., Milgram, R. J., Schmid, W., & Schaar, R. (2005). Reaching for common ground in K – 12 Mathematics Education. *Notices of the AMS*, *52*(9), 1055–1058.
Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, *32*(3), 267–295. https://doi.org/10.2307/749828
Mitchell, A., & Clarke, D. M. (2004). When is three quarters not three quarters? Listening for conceptual understanding in children's explanations in a fractions interview. In I. Putt, R. Farragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010 (proceedings of the 27th annual conference of the mathematics education research group of Australasia)* (pp. 367–373). MERGA.
Moss, J. (2005). Pipes, tubes, and beakers: New approaches to teaching the rational-number system. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: Mathematics in the classroom* (pp. 121–162). National Academies Press.
Murphy, G. (2004). *The big book of concepts*. MIT Press.
Niss, M. (2006). The structure of mathematics and its influence on the learning process. In J. Maasz & W. Schloeglmann (Eds.), *New mathematical education research and practice* (pp. 51–62). Sense Publishers.
Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *Journal of Mathematical Behavior*, *28*(2), 150–161. https://doi.org/10.1016/j.jmathb.2009.06.002
O'Halloran, K. L. (2005). *Mathematical discourse: Language, symbolism and visual images*. Continum.
Olive, S. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning*, *1*(4), 279–314. https://doi.org/10.1207/s15327833mtl0104_2
Olsson, J., & Granberg, C. (2024). Teacher-student interaction supporting students' creative mathematical reasoning during problem solving using Scratch. *Mathematical Thinking and Learning*, *26*(3), 278–305. https://doi.org/10.1080/10986065.2022.2105567
Pedersen, P. L., & Bjerre, M. (2021). Two conceptions of fraction equivalence. *Educational Studies in Mathematics*, *107*(1), 135–157. https://doi.org/10.1007/s10649-021-10030-7
Pepin, B. (2012). Task analysis as a "catalytic tool" for feedback and teacher learning: Working with teachers on mathematics curriculum materials. In G. Guedets, B. Pipin, & L. Trouche (Eds.), *From text to 'lived' resources* (pp. 123–142). Springer.
Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, *23*(6), 595–610. https://doi.org/10.1007/BF00540062
Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, *23*(2), 139–159. https://doi.org/10.1080/10573560601158461
Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press.
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36. https://doi.org/10.1007/BF00302715
Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge University Press.
SFS 2003:460. *The Ethics Review Act*. https://www.riksdagen.se/sv/dokument-och-lagar/dokument/svensk-forfattningssamling/lag-2003460-om-etikprovning-av-forskning-som_sfs-2003-460/

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–697. https://doi.org/10.1177/0956797612440101

Simon, M. A. (2017). Explicating "mathematical concept" and "mathematical conception" as theoretical constructs for mathematics education research. *Educational Studies in Mathematics*, *94*(2), 117–137. https://doi.org/10.1007/s10649-016-9728-1

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91–104. https://doi.org/10.1207/s15327833mtl0602_2

Skolverket. (2022). *Läroplan för grundskolan, förskoleklassen och fritidshemmet* 2022. *Skolverket*.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503–518. https://doi.org/10.1016/j.learninstruc.2004.06.015

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection. From research to practice. *Mathematics Teaching in the Middle School*, *3*(4), 268–275. https://doi.org/10.5951/MTMS.3.4.0268

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education*, *18*(2), 123–140. https://doi.org/10.1007/s10857-014-9279-2

Swedish Research Council. (2017). Good research practice. Swedish Research Council.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. https://doi.org/10.1007/BF00305619

Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, *37*, 5–13. https://doi.org/10.1016/j.learninstruc.2014.03.002

Usiskin, Z. (2018). Electronic vs. Paper textbook presentations of the various aspects of mathematics. *ZDM Mathematics Education*, *50*(5), 849–861. https://doi.org/10.1007/s11858-018-0936-2

Vamvakoussi, X. (2015). The development of rational number knowledge: Old topic, new insights. *Learning and Instruction*, *37*, 50–55. https://doi.org/10.1016/j.learninstruc.2015.01.002

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*(1), 9–35. https://doi.org/10.1023/B:EDUC.0000005212.03219.dc

Van Dooren, W., Lehtinen, E., & Verschaffel, L. (2015). Unraveling the gap between natural and rational numbers. *Learning and Instruction*, *37*, 1–4. https://doi.org/10.1016/j.learninstruc.2015.01.001

Vig, R., Murray, E., & Star, J. R. (2014). Model breaking points conceptualized. *Educational Psychology Review*, *26*(1), 73–90. https://doi.org/10.1007/s10648-014-9254-6

Vinner, S. (2020). Concept development in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 123–127). Springer.

Viseu, F., Pires, A. L., Menezes, L., & Costa, A. M. (2021). Semiotic representations in the learning of rational numbers by 2nd grade Portuguese students. *International Electronic Journal of Elementary Education*, *13*(5), 611–624. https://doi.org/10.26822/iejee.2021.216

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematics Thinking and Learning*, *8*(2), 91–111. https://doi.org/10.1207/s15327833mtl0802_1

Wedman, L. (2020). *The concept concept in mathematics education: A concept analysis* (Gothenburg Studies in Educational Science, 450) [Dissertation]. Acta Universitatis Gothoburgensis.

Wright, V. (2014). Towards a hypothetical learning trajectory for rational number. *Mathematics Education Research Journal*, *26*(3), 635–657. https://doi.org/10.1007/s13394-014-0117-8

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, *60*(3), 297–321. https://doi.org/10.1007/s10649-005-0606-5